

Parameter Inversion Model for Two Dimensional Parabolic Equation Using Levenberg-Marquardt Method

Tao Min^{1,*}, Zhulin Hao¹, Minquan Feng¹, Hansong Tang²

1. Xi'an University of Technology, Xi'an 710054, China

2. Dept. of Civil Eng., City College, The City University of New York, New York 10031

Abstract: The problem of determining unknown parameters in the two-dimensional heat equation is considered. A method based on the Levenberg–Marquardt algorithm (LMA) is examined. The approach is successfully applied to solve the inverse problem of a two-dimensional parabolic equation whose coefficient is a partition paragraph function. The numerical results demonstrate the effectiveness of the proposed method.

Keywords: parabolic equation; inverse problem; Levenberg–Marquardt

0. Introduction

Inverse parabolic problems arise from various backgrounds in engineering. For instance, in thermal systems, thermal properties, including the heat convection coefficient (Martin et al., 1998) and temperature dependent thermal conductivity (Dowding et al., 1999), are often unknown and need to be recovered. Recent applications also include the optical, wherein the interest is to recover anomalies in human tissues (Barbour et al., 1995. National Research Council, 1996). In most applications, one is led to excite the system by an external means and record the associated system response. The collected data are then used to recover the sought-after unknown. These kinds of problem have been investigated by many researchers. Recent results include an analytical method for the solution of the over-determined inverse heat conduction (Taler, 1997), application of neural networks for the recovering of electrical conductivity profiles (Glorieux et al., 1999), a spectral method for solving the lateral heat equations (Berntsson, 1999), and a discrete diffusive model for the recovery of the absorption coefficient from diffused reflected light (Martiz et al., 1998). Additional methods include nonlinear optimization using genetic algorithms (Stoffa et

* Correspondence: mintao@xaut.edu.cn

al., 1991), Marquardt's procedure, and thermal wave- slice tomography. Most of the discussions in the literature are devoted to the qualitative analysis of the equations such as existence and uniqueness of solution.

1. The Problem

In this paper, we study numerical procedures for the inverse problem of simultaneously finding the unknown functions $p(x, y)$ and $u(x, y, t)$ to satisfy:

$$\frac{\partial u}{\partial t} + a(x, y) \frac{\partial u}{\partial x} + b(x, y) \frac{\partial u}{\partial y} - \frac{\partial}{\partial x} \left[p(x, y) \frac{\partial u}{\partial x} \right] - \frac{\partial}{\partial y} \left[p(x, y) \frac{\partial u}{\partial y} \right] - d(x, y)u = f(x, y, t) \quad 0 < t < T \quad (1)$$

$$(x, y) \in \Omega$$

with the initial-boundary conditions,

$$u(x, y, 0) = u_0(x, y) \quad (2)$$

$$\frac{\partial u(x, y, t)}{\partial n} + \lambda u(x, y, t) = \beta \quad (x, y) \in \Gamma \quad (3)$$

and the additional condition,

$$u(x, y, t) = u_\eta(x, y, t), \quad x, y \in \Gamma_\eta \subset \Gamma \quad (4)$$

where $a(x, y)$, $b(x, y)$, $d(x, y)$, $f(x, y, t)$, and $u_\eta(x, y, t)$ are known functions,

λ and β are given parameters, and the functions $u(x, y, t)$ and $p(x, y)$ are unknown. When

$p(x, y)$ is given, there is a rather satisfactory theory for the direct problem (1)–(3) regarding

posedness and other related properties. When $p(x, y)$ is unknown, in order to find a solution as

well as $p(x, y)$, one needs additional condition (4), the actual form of which depends on the

concrete physical model.

2. Nonlinear Optimization and Levenberg–Marquardt Algorithm

In general, β and $u_\eta(x, y, t)$ are known in the fixed, prescribed interior point in Ω whose

boundary is denoted by $\partial\Omega$. Assuming that T is the sampling period, they can be measured at

$t = iT (i = 0, 1, \dots, I)$. Further assume that $p^*(x, y)$ is the exact solution of $p(x, y)$ and

$u^*(x, y, t)$ is the exact solution to the problem (1)–(3). Let K be a complete linear space of real numbers and $p^*(x, y) \in K$, then $\Phi_1(x, y), \Phi_2(x, y), \dots, \Phi_n(x, y)$ is a group of basis functions in K .

Then

$$p^*(x, y) = \sum_{i=1}^{\infty} k_i^* \Phi_i(x, y) \tag{5}$$

We represent $p^*(x, y)$ as a finite sum of the form

$$p^*(x, y) \approx \sum_{i=1}^n k_i^* \Phi_i(x, y) \tag{6}$$

the size of n is determined by approximate accuracy.

Therefore solving the inverse problems is to determine an n -dimensional real vector

$$\mu = K^T = (k_1, k_2, \dots, k_n) \in R^n \tag{7}$$

Then the function

$$p(x, y) = \sum_{i=1}^n k_i \Phi_i(x, y) = K^T \Phi(x, y)$$

satisfies the model (1) ~ (4), where $\Phi(x, y) = (\Phi_1(x, y), \Phi_2(x, y), \dots, \Phi_n(x, y))^T$.

If $u(p(x, y); x, y, t)$ is the solution of the initial boundary value problem (1) ~ (3) corresponding to $p(x, y)$, the problem of determining $p(x, y)$ can be converted to find the solution of the following minimization problem:

$$\min \varphi = \left\| u_{\eta}^*(x, y, t) - u(p(x, y); x, y, t) \right\|_{(x, y, t) \in \Gamma_{\eta}}^2 \tag{8}$$

where $\Gamma_{\eta} = \{(x, y, t) \in \Gamma_{\eta}, t \in [0, iT]\} (i = 0, 1, \dots, I)$ and T is the sampling period of $u_{\eta}^*(x, y, t)$. From (7) we know we can find a n -dimensional real vector μ which minimizes the functions φ . Then instead of (8), we study the problem:

$$\min \varphi = \sum_{j=1}^m \sum_{i=0}^I (u_{\eta}^*(x_j, y_j, t_i) - u(\mu; x_j, y_j, t_i))^2 \tag{9}$$

where $u(\mu; x, y, t)$ satisfies the model (1) ~ (4).

Multiplying both sides of (9) by 0.5 produces a general non-linear least squares problem:

$$\min f = \frac{1}{2} \sum_{l=1}^{m(I+1)} [r_l(\mu)]^2 \quad (10)$$

where $f = \frac{1}{2} \varphi$; $r_l(\mu) = u_{\eta}^*(x_j, y_j, t_i) - u(\mu; x_j, y_j, t_i), (i = 0, 1, \dots, I, j = 1, 2, \dots, m)$.

The inverse problem is thus converted into a nonlinear optimization problem. Let us introduce the Levenberg–Marquardt algorithm as follows:

Initialize: $k=0$,

μ^0 : initial guess vector of μ

At iteration k :
$$\mu^{(k+1)} = \mu^{(k)} + \delta^{(k)}, k = 0, 1, 2, \dots \quad (11)$$

$\delta^{(k)}$ can be solved by a set of linear equations:

$$(A_k A_k^T + \mu I) \delta = -A_k r^{(k)} \quad (12)$$

where: $\mu \geq 0$, $A(\mu) = [\nabla r_1(\mu), \dots, \nabla r_{m(I+1)}(\mu)]$, and $\nabla r_i(\mu) = \frac{\partial r_i}{\partial \mu}$. Solving the above

equations, a new initial guess vector is obtained when we substitute its solution $\delta^{(k)}$ into (11).

The above solution process is repeated until the data meet the accuracy requirements. In the above-mentioned algorithms, each iteration should solve the direct problem. The direct problem involved in this article is being implemented with a finite element method.

3. Numerical Examples

Letting $p^*(x, y)$ be the exact solution, we compute $u(p^*(x, y); x, y, iT)$ by solving the direct problem, which yields the additional data recorded as $u_{\eta}; \hat{p}(x, y)$ is obtained by the above-mentioned algorithm. We then compare $\hat{p}(x, y)$ with the true solution $p^*(x, y)$.

Example 1

Consider problem (1) ~ (4) on $\Omega \in [0, 1] \times [0, 1]$ with the following conditions:

$$a(x, y) = t + y, b(x, y) = t + x, d(x, y) = t + x + y$$

$$f(x, y, t) = -5e^{x+y} + (t+y)((x+y+t)e^{x+y} + e^{x+y}) + (t+x)((x+y+t)e^{x+y} + e^{x+y}) - 6(x+y+t)e^{x+y} - 2(3x+3y+5t)((x+y+t)e^{x+y} + 2e^{x+y}) + (x+y+t)^2 e^{x+y}$$

with the initial-boundary conditions

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega$$

$$\begin{aligned} -\frac{\partial u}{\partial x}\Big|_{x=0} &= -e^y(y+t) - e^y & \frac{\partial u}{\partial x}\Big|_{x=1} &= e^{1+y}(1+y+t) + e^{1+y} & (0 \leq y \leq 1, t > 0) \\ -\frac{\partial u}{\partial y}\Big|_{y=0} &= -e^x(x+t) - e^x & \frac{\partial u}{\partial y}\Big|_{y=1} &= e^{x+1}(x+1+t) + e^{x+1} & (0 \leq x \leq 1, t > 0) \end{aligned}$$

and the additional condition

$$u(x, y, t)\Big|_{x=1} = \phi(y, t)$$

the time sampling is taken as $i, j = 1, 2, \dots, n$ $t = 0.1, t = 0.3, t = 0.5$, i.e. $I = 3$.

The determined parameter is $p(x, y) = k_1x + k_2y + k_3t$, where $k_i (i = 1, 2, 3)$ is to be determined. The true solution is $p^*(x, y) = 3x + 3y + 5t$, the direct problem is being implemented with the finite element method. The initial value is taken as $p_0(x, y) = x + y + t$ and $p_0(x, y) = 2x + 2y + 2t$, the results are shown in Table 1:

Table.1 Numerical simulation results of parameter

parameter	$K_0 = [1, 1, 1]$			$K_0 = [2, 2, 2]$			The true solution
	Iterations	2	4	6	2	3	
k_1	2.6171	2.9979	2.6607	2.9616	3.0005	3.0002	3
k_2	3.2780	3.1712	4.8296	3.2185	2.8612	2.9999	3
k_3	2.6607	3.0024	4.9980	4.4260	5.0878	5.0001	5
Error	2.3867	0.2416	0.0031	0.6154	0.1642	2.4495×10^{-4}	

The error between the numerical solution $K = [k_1, k_2, k_3]$ and the true solution

$$K^* = [k_1^*, k_2^*, k_3^*] \text{ is } ER = \sqrt{(k_1 - k_1^*)^2 + (k_2 - k_2^*)^2 + (k_3 - k_3^*)^2}.$$

The error $ER = \sqrt{(k_1 - k_1^*)^2 + (k_2 - k_2^*)^2 + (k_3 - k_3^*)^2}$ between the numerical solution $K = [k_1, k_2, k_3]$

and the true solution $K^* = [k_1^*, k_2^*, k_3^*]$ for iteration L is shown in Figure 1:

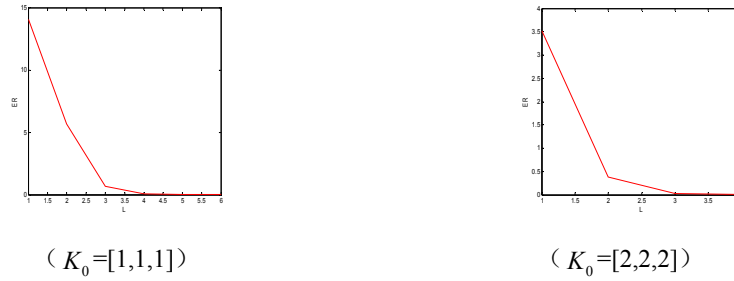


Figure.1 The relationship between the computation accuracy and the number of iterations.

In the following section, the data in Example 1 were modified by random perturbations (errors) of 1%, 5% and 10%:

$$u_{\eta}^{\delta} = u_{\eta} (1 + \theta\delta)$$

where: θ = the random variable between $[-1, 1]$; $\delta = 0.01, 0.05, \text{ and } 0.1$, respectively. For the different initial guess and different random measurement error, simulation results are shown in Table2:

Table.2

Noise $r\%$	1		5		10	
Initial guess K_0	(1,1,1)	(2,2,2)	(1,1,1)	(2,2,2)	(1,1,1)	(2,2,2)
Iterations	7	5	6	6	6	6
The resulting value	2.9602	2.9601	2.9045	2.9310	2.6263	2.8621
	3.0370	3.0367	3.1316	3.1245	3.3728	3.2528
	5.0001	5.0003	4.9851	4.9626	5.0354	4.9221
ER	0.0543	0.0542	0.1633	0.1472	0.5290	0.2983

The error between the numerical solution $K=[k_1, k_2, k_3]$ and the true solution

$$K^*=[k_1^*, k_2^*, k_3^*] \text{ is } ER = \sqrt{(k_1 - k_1^*)^2 + (k_2 - k_2^*)^2 + (k_3 - k_3^*)^2}.$$

Example 2

In order to demonstrate the performance of the proposed scheme in dealing with discontinuous coefficients in the equations, consider problem (1)-(4) on $\Omega \in [0,1] \times [0,1]$ and subject to the following conditions:

$$a(x, y) = x + 2y, b(x, y) = 2x + y, p(x, y) = x^2 + y$$

$$f(x, y, t) = ye^{x+t} - 2xye^{x+t} - (x^2 + 1)ye^{x+t} + (1 + x + y)^2 ye^{x+t}$$

with the initial-boundary conditions

$$u(x, y, 0) = e^x y$$

$$\frac{\partial u}{\partial n} = e^{x+t} y$$

and the additional condition

$$u(x, y, t)|_{x=1} = \phi(y, t)$$

the time sampling is taken as $i, j = 1, 2, \dots, n$ $t = 0.1, t = 0.3, t = 0.5$, i.e. $I = 3$.

The determined parameter is $d(x, y) = \begin{cases} k_1 x + k_2 y & 0 < x < 1, \quad 0 < y < 1/2 \\ k_3 & 0 < x < 1, \quad 1/2 \leq y < 1 \end{cases}$, where

$k_i (i = 1, 2, 3)$ is to be determined. The true solution is a piecewise function

$$d^*(x, y) = \begin{cases} 2x + 4y & 0 < x < 1, \quad 0 < y < 1/2 \\ 4 & 0 < x < 1, \quad 1/2 \leq y < 1 \end{cases}$$
 the direct problem is being implemented with the

finite element method. The initial value is taken as $d_0(x, y) = \begin{cases} 0.5x + 0.5y & 0 < x < 1, \quad 0 < y < 1/2 \\ 0.5 & 0 < x < 1, \quad 1/2 \leq y < 1 \end{cases}$

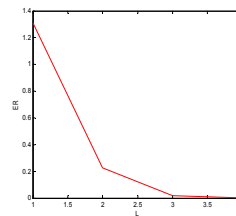
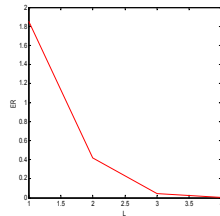
and $d_0(x, y) = \begin{cases} x + y & 0 < x < 1, \quad 0 < y < 1/2 \\ 1 & 0 < x < 1, \quad 1/2 \leq y < 1 \end{cases}$, the results are shown in Tables 3:

Table. 3 Numerical simulation results of parameter ($K_0 = [0.5, 0.5, 0.5]$)

parameter	$K_0 = [0.5, 0.5, 0.5]$			$K_0 = [1, 1, 1]$			The true solution
	Iterations	2	3	4	2	3	
k_1	1.7389	1.9805	1.9998	1.8780	1.9932	1.9999	2
k_2	4.1899	4.0358	4.0004	4.1245	4.0163	4.0002	4
k_3	4.1899	3.9896	3.9999	3.8545	3.9963	4.0000	4
Error	0.4198	0.0421	4.5826×10^{-4}	0.2271	0.0180	2.2361×10^{-4}	

The error $ER = \sqrt{(k_1 - k_1^*)^2 + (k_2 - k_2^*)^2 + (k_3 - k_3^*)^2}$ between the numerical solution $K = [k_1, k_2, k_3]$

and the true solution $K^* = [k_1^*, k_2^*, k_3^*]$ for iteration L is shown in Figure 2:



$$(K_0=[0.5,0.5,0.5])$$

$$(K_0=[1,1,1])$$

4. Conclusion

The results illustrate that the Levenberg–Marquardt method is applicable and efficient in the case of determination of an unknown parameter in the two-dimensional heat equation. In particular, the algorithm was steady for data with a random disturbance. The method is successfully applied the inverse problem of a two-dimensional parabolic equation whose coefficient is a partition paragraph function. It turns out that the best perturbation method is one of the efficient methods to solve this kind of problems.

Its rate of convergence is high and its domain of convergence is wide enough to be successfully used in practice. Note that the algorithm has a certain dependence on the initial guess of the inverse parameter.

References

- T.J. Martin, Dulikravich, Inverse determination of steady heat convection coefficient distribution, *Journal of Heat Transfer* 120, 328-334, 1998.
- K.J. Dowding, J.V. Beck and B.F. Blackwell, Estimating temperature-dependent thermal properties, *Journal of Thermophysics and Heat Transfer* 13 (3), 328-336, 1999.
- R. Barbour, H. Graber, J. Chang, S. Barbour, S. Koo and R. Aronson, MRI guided optical tomography, *IEEE Comp. Sci. Engng.* 2 (4), 63-77, 1995.
- National Research Council, *Mathematics and Physics of Emerging Biomedical Imaging*, National Academic Press, Washington, DC, 1996.
- J. Taler, Analytical solution of the overdetermined inverse heat conduction problem with an application to monitoring thermal stresses, *Heat and Mass Transfer* 33, 209-218, 1997.
- C. Glorieux, J. Moulder, J. Basart and J. Thoen, The determination of electrical conductivity profiles using neural networks inversion of multi-frequency eddy-current data, *Journal of Physics D: Appl. Phys.* 32, 612-622, 1999.
- F. Berntsson, A spectral method for solving the sideways heat equation, *Inverse Problems* 15, 891-906, 1999.
- M.F. Martiz, G.T. Herman and C. Yee, Recovery of the absorption coefficient from diffused reflected light using a discrete diffusive model, *SIAM Journal on Applied Mathematics* 59, 58-71, 1998.
- P.L. Stoffa and M.K. Sen, Nonlinear multiparameter optimization using genetic algorithms: Inversion of plane-wave seismograms, *Geophysics* 56, 1794-1810, 1991.